Catherine's Strategy and Transition from Concrete Representations to More Abstract Thinking While Solving Addition Story Problems

This story is a part of the series:

What's Next? **Stories of Teachers Engaging in Collaborative Inquiry Focused on Using Student Thinking to Inform Instructional Decisions**

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Stories of teachers engaging in collaborative inquiry focused on using student thinking to inform instructional decisions

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Introduction

A group of teachers explored first-grade students' understanding of place value and base-ten concepts through addition word problems. They planned and carried out a lesson that addressed the various ways of adding two numbers with emphasis on place-value concepts. Emphasis was also placed on ways to count and how to express strategies in writing. This narrative offers insight into students' emerging understanding of numbers and highlights ways to connect less sophisticated (e.g., concrete) strategies with more sophisticated (e.g., abstract) strategies.

Relevant Florida Mathematics Standards

MAFS.1.NBT.2.2 Understand that the two digits of a two-digit number represent amounts of tens and ones.

a. 10 can be thought of as a bundle of ten ones—called a "ten."
b. The numbers from 11 to 19 are composed of a ten and one, two, three, four, five, six, seven, eight, or nine ones.
c. The numbers 10, 20, 30, 40, 50, 60, 70, 80, or 90 refer to one, two, three, four, five, six, seven, eight, or nine tens (and 0 ones).
d. Decompose two-digit numbers in multiple ways (e.g., 64 can be decomposed into 6 tens and 4 ones or into 5 tens and 14 ones).

MAFS.1.NBT.3.4 Add within 100, including adding a two-digit number and a one-digit number and adding a two-digit number and a multiple of 10, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used. Understand that in adding two-digit numbers, one adds tens and tens, ones and ones, and sometimes it is necessary to compose a ten.

Background Information

To understand better the mathematical concepts used in this lesson, consider reading chapters seven and nine in *Children's Mathematics: Cognitively Guided Instruction* (Carpenter et al., 2015). Chapter seven provides background on children's understanding of place-value and base-ten concepts. It also offers a research-based development of student thinking toward a more complex use of base-ten concepts in strategies. Chapter nine discusses classroom environments designed to foster the development of understanding of base-ten concepts and how to elicit student thinking to extend mathematical ideas.

Carpenter, T. P, Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (2015) *Children's Mathematics: Cognitively Guided Instruction*. Portsmouth, NH. Heinemann.

Analyzing Student Thinking

As part of a professional-development program, a group of teachers studied how the understanding of place-value concepts and the base-ten number system emerge in first-grade students. As part of this investigation, the teachers conducted brief interviews with individual students in a first-grade classroom about midway through the school year. They used the observations gained in these interviews to create and implement a lesson designed to extend the first-graders' mathematical ideas on that same day. The one-on-one interviews, which each lasted approximately 20 minutes, were intended to assess student thinking. Teachers did not provide feedback to students about their solution strategies, but they did encourage students to explain their thinking. The teachers posed the following set of four addition word problems, one at a time, during the interviews.

> *Problem A. Pete has 20 rocks. Juan gives him 24 more rocks. How many rocks does Pete have now?*

> *Problem B. Tylesha has 32 books. Her grandma gives her 25 more books. How many books does Tylesha have now?*

Problem C. Mr. Jones had 40 cupcakes. He gave 20 cupcakes to the students in his class. How many cupcakes does Mr. Jones have now?

Problem D. Maria had 35 jellybeans. Her dad gave her 27 more jellybeans. How many jellybeans does Maria have now?

The students were encouraged to use their own strategies to solve these problems using either manipulatives or mental strategies. Base-ten blocks and paper and pens were available to each interviewee. The teachers also took detailed notes on the interviewed students' thinking and solution path used to solve each problem.

Immediately after the interviews, the teachers reflected on the strategies the first-grade students used. They paid particular attention to the use of place value and any understanding of base-ten concepts evident in the student solutions. The teachers organized the students into the following categories on the basis of the most sophisticated strategy they used during the interview.[1]

Multidigit Computation Strategies

A student who uses a *direct modeling with ones* strategy represents each multidigit number in the problem as a set of ones using manipulatives or pictures to model the story in the problem and then counts the objects or pictures to determine the answer. For example, a student using this strategy for the Pete's-rocks problem might first create a set of 20 objects, then create another set of 24 objects, combine this second set with the first set, and finally count the combined set by ones.

A student who uses a *counting by ones* strategy does not physically represent every quantity in the problem. Fingers, objects, or tally marks are often used to keep track of the number of counts. For the Tylesha's-books problem, for example, a student using a *counting by ones* strategy might say "32" and then proceed to count forward by ones 25 times while keeping track of the count with fingers.

A student who uses a *directing modeling with tens* strategy represents each multidigit number in the problem using manipulatives or pictures that reflect the base-ten structure of the number system (e.g., with base-ten blocks or base-ten pictures). Then, the student counts the objects or pictures by tens and ones to determine the answer. In the Mr. Jones'-cupcakes problem, for example, the student could use base-ten blocks to model both 40 and 20 by representing the numbers using four ten rods and two ten rods, respectively. The student might then count the number of rods by tens. If a problem—such as the cupcake problem—has nondecade numbers (32 and 25), the student may count first the tens and then the ones saying, "10, 20, 30, 40, 50, 51, 52, 53, 54, 55, 56, 57."

A student who uses an *invented algorithms* strategy demonstrates flexibility in thinking about numbers by breaking them apart and putting them together in different ways without the use of manipulatives or pictorial models. Three commonly used invented algorithms are incrementing, combining the same units, and compensation.

A student who uses *combining the same units* operates on the tens and ones separately and the partial sums are then combined to get a final result. In the Maria's-jellybeans problem the student might add the three tens from 35 and the two tens from 27 to get 50, might add the five and the seven to get 12, then might add 50 and 12 together.

A student using an *incrementing* strategy determines the answer by incrementing or decrementing partial sums or differences. For exam-

1 The descriptions of strategies presented here are the current descriptions used by our team, and we consider them fluid, as our understanding of these ideas continues to evolve. For a more detailed discussion of these terms, consider reading Carpenter et al. (2015).

ple, a student using this strategy to solve the Maria's-jellybeans problem might add 27 onto 35 in increments that lead to decade numbers (27 = 5 + 20 + 2). So, the student might increment 35 by five to get to 40, then add 20 getting 60, and finally add the remaining two.

A student who uses a *compensation* strategy adds or subtracts the numbers by adjusting one number to compensate for changes made in another number. To solve the Maria's-jellybeans problem, a student using this strategy might think it easier to add 30 to 35 and then adjust the sum by subtracting three because he or she added too much.

Strategies used by students in this classroom

The first-grade students were classified according to the strategies described in the previous section in accordance with the most sophisticated strategy they used to solve any of the four problems during interviews. Figure 1 below displays how the students were originally categorized as well as the final consensus on what their strategies were and on which of their strategies was the most sophisticated.

Initially, the teachers had categorized many of the students as having used a *counting by ones* strategy as their most sophisticated way to solve an addition problem. The teachers mislabeled some students as having used the *counting by ones* strategy for the Pete's-rocks problem. Those students modeled both the 20 and 24 sets of rocks using individual items and combined the two sets to find the answer, but counted the entire items starting from 20. Specifically, they referred to the first set of objects as "20" and kept counting, "21, 22, 23, 24, …, 44." A brief review of the descriptions of the various strategies persuaded the teachers that these students had, in fact, used a *direct modeling with ones* strategy, because they represented each number with sets of individual objects.

The decision of which strategies were more sophisticated than others was agreed upon after some discussion and revisiting of the research on the conceptual understanding of student thinking (found in Carpenter et al., 2015). The teachers considered the *direct modeling with ones* strategy to be the least sophisticated of the ways a multidigit addition problem can be solved, because students did not use base-ten concepts and used objects or pictures to model the story in the problem. The *counting by ones* strategy was considered the second least sophisticated strategy in the context of the interview problems, because the students did not use base-ten concepts and combined the second quantity with the first quantity by counting on by ones. This way of adding was more

Direct modeling with ones	Direct modeling with tens	Counting by ones	Invented algorithms		
			Combine like units	Incrementing	Compensation
Madison S.	Stephen	Nicole	Gianluca	Catherine*	
Drew	Timothy	Brayan			
My'on	Samantha J.				
Trinity	Raina				
Adrian	Ella				
Noah					
Midori					
Madison C.					

* Catherine's strategy was not a standard *incremental* strategy.

Figure 1. Classification of the interviewed students based on strategies used.

time consuming and error-prone because both numbers added were double digit. *Direct modeling with tens* was considered to be a more sophisticated strategy, because the students demonstrated an understanding of base-ten concepts while representing the story problem. The *invented algorithms* were considered the most sophisticated strategies of all, because students made use of mental structures and did not need to use tools (e.g. manipulatives, fingers) to represent the story. The three types of *invented algorithm* strategies were considered equally sophisticated.

The teachers observed that the most sophisticated strategies were used when students were solving the Pete's-rocks problem. Pete's rocks had the smallest numbers of the problems set. After sorting students according to the strategies they used, the teachers noticed that a large group of first graders used *direct modeling with ones*, a smaller group used *direct modeling with tens*, and two students used a *counting by ones* strategy. Only one student—Gianluca—used a standard *invented algorithm* strategy. Gianluca solved the books problem by the *combining like units* strategy. He decomposed the given numbers by place value, mentally added the quantities in the tens (30 +20) and then the ones (2 + 5) places, and then wrote 50 + 7 = 57 on paper. Catherine's strategy was not a standard *incrementing* strategy, because she had used some physical tools to aid her thinking.

The teachers were intrigued by Catherine's strategy for solving the Pete's-rocks problem. They spent a significant amount of time analyzing it. Catherine's strategy was labeled as an *incrementing* strategy, but the teachers pondered whether it might be classified as a *direct modeling with tens* strategy, because she used base-ten blocks. The actions in her strategy did not parallel the story as it unfolds in the word problem. Instead, she started by modeling 24 and then adding 20 to it in her mind. Moreover, she only used base-ten rods to model 24, using three base-ten rods and covering six units on one of the rods. She did not pull out additional manipulatives to model the other number, but instead said, "24 and ten is 34, 34 and ten more is 44." The teachers considered she may have used mental representations of the latter two rods of ten. They finally decided Catherine's strategy seemed to represent a blend between a *direct modeling* and an *incrementing* strategy. On paper, however, she drew out the models for both 24 and 20 using connected units grouped in tens and gave the answer, 44. The teachers acknowledged that Catherine's way of solving the problem did not match her written explanation and that the written work missed the sophistication of her thinking.

On the basis of the students' strategies on the interview problems, the teachers set a number of learning goals to extend the students' understanding of the base-ten number system:

1. *Students who direct modeled with ones will understand how groups of ten can be used in direct modeling strategies.*
2. *Students will use more efficient/sophisticated strategies.*
3. *Students' written records of their strategies will match the actual strategies they used.*

Planning for the Lesson

In light of the selected goals, the teachers next developed a lesson around a discussion of students' interview strategies to address ways to be more efficient in thinking about numbers, in counting large groups of objects, and in recording their strategies. They planned to use student strategies from the interview—specifically, from the Pete's-rocks problem—to start the scaffolding of knowledge about place value and efficient counting. They developed the following problem to use in the latter part of the lesson.

> *I had 28 lollipops. My friend gave me 9 more. How many lollipops do I have*

now?

The problem and the numbers in it were designed to be used at the end of the lesson to give students the opportunity to engage with concepts they would encounter during the discussion of the strategies students used earlier in the lesson.

Plan for the strategies discussion

The observations the teachers made about the first-graders' thinking helped them understand more about the students' understanding of place-value concepts. They thought that engaging students in a discussion about different strategies could benefit the largest group of students in the classroom, who did not use strategies involving groups of tens at all. Such a discussion could also benefit the students who started using tens by extending their understanding of place value, efficient counting, and mathematically accurate notations. In addition, highlighting the way that some students counted by tens might help their peers to understand how place value concepts can help reduce the work involved in solving the problem (as compared with counting by ones). Having noticed that the most sophisticated strategies were used with the Pete's-rocks problem, the teachers planned to use this problem to build a whole-class discussion about the various ways students used tens to add two multidigit numbers.

The extensive discussion of Catherine's strategy led to a conjecture that *direct modeling* strategies lead to the use of abstract *invented algorithms*, but only after the children develop a sufficient amount of knowledge of numbers—or trust in their ability to use them—that helps them move on from representing the story with physical objects or drawings. They therefore agreed on the importance of a discussion of the full range of student strategies used in the class and highlighting the connections among them. An obstacle to the transition from using visual or concrete representations to mental strategies, the teachers decided, was the inability to retain the first number in working memory. They conjectured that this obstacle could be turned into a stepping stone toward a more complex working understanding of quantities if the class discussion emphasized retaining the cardinality of a set in the students' minds without physical representation of the set of individual objects. This advance could be achieved, they considered, by a focus of part of the discussion during the lesson on a *counting* strategy, an *invented algorithm* strategy, and Catherine's work. They considered the discussion of these strategies to be working on the second chosen learning goal.

To address the third goal of students' accurate recording of their strategies on paper, the teachers determined to use student work that most closely matched the students' actual strategies. They looked for clear recording of drawn models and written equations on students' interview papers. They also

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but only after the children develop a sufficient amount of knowledge of numbers—or trust in their ability to use them—that helps them move on from representing the story with physical objects or drawings.

planned to ask questions during the lesson that addressed the accuracy and clearness of the written work and for the instructor to connect physical models to numbers and equations as the students explained what the models represented. If needed, the instructor would add numbers and equations to the interview work sheets to make them clearer. The teachers also considered having the instructor write the equations that matched the strategies on the board as the students explained their thinking. Seeing how a student's thinking is translated to a mathematical sentence in this way might solidify the class' understanding of mathematical notations and equations.

The teachers next selected the students whose strategies would be most likely to create an interesting discussion about the different ways to solve the Pete's-rocks problem. They decided to order the student presentations from what the teachers considered to be the least sophisticated to most sophisticated strategy, which corresponds to moving from left to right in Table 1. Because the list of strategies they wished to discuss was so large, the teachers decided not to delve deeply into each strategy. Instead, they chose to draw connections across all the strategies by focusing on numbers and counting. The lesson's instructor would have the written student work from the interviews prepared to be displayed in class during the lesson. The teachers determined the following students would share, in the following order:

1. Trinity was selected to share first because she used base-ten rods and counted one of the numbers by ten. A discussion about more efficient ways of counting was planned for this segment of the discussion.

2. Timothy modeled the base-ten rods without drawing the individual elements and made a clear record of his strategy.

3. Nicole held the first number in her head and counted on to find the answer. This would be a good way to introduce the concept of solving a problem without having to use concrete objects to model the quantities. A discussion of how to record this strategy was planned for this segment of the lesson.

4. Gianluca used *combining like units* on most problems. On the Pete's-rocks problem, his strategy could be interpreted as more of an *incrementing* strategy, because he added 20 (which did not have ones to separate from tens) in chunks. His work would be a great way to challenge the more advanced thinkers (those who used *counting* and *direct modeling with tens* strategies).

5. Catherine was selected because of her use of *incrementing* addition and because of her flexible choice of starting number. Her starting with 24 might provide an "aha" moment for some students and encourage the use of more efficient ways to solve story problems.

Rationale for the selected problem

The teachers also created a new problem for students to solve that would give them an opportunity to experiment with any new ideas they encountered in the discussion. In order to focus on place-value concepts, the teachers chose a simple structure (i.e., join change unknown) so that students could easily interpret the problem and focus their attention on the numbers. They chose a context the students would be familiar with (lollipops), because the students in this class often talked about their fondness for these sweets.

The teachers engaged in a long discussion about what numbers to use in the problem to address the chosen goals of the lesson for these students. The interview notes suggested that several students were not able to count or solve the problems if the numbers were too high. The teachers decided to keep the sum less than 40 and one of the numbers around 30 to ensure the numbers themselves would not be an obstacle to students' successful engagement with the problem while still providing opportunities for students' place-value ideas to develop.

The benefits and drawbacks of several number pairs were considered for the lollipop problem, careful consideration was given to the strategies each of these number pairs would encourage. The first pair of numbers pondered was (25, 4). The teachers thought 25 would be a good number to model using tens and ones. It would also be an easy number set for students using *counting* strategies, keeping 25 in their heads and counting on four more. To solve using *invented* algorithms, a student might think of using five instead of four and then adjust the answer, 25 + 5 =30, 30 – 1 = 29. This number combination (i.e., 25, 4) would not be likely to result in students' using *combining like units* strategies, because the second number is less than ten. The (20, 16) pair could be useful in moving students to *direct modeling with tens* strategy. It could also be solved by an *incrementing* strategy, such as 20 + 10 = 30, 30 + 6 = 36. This set of numbers would not be likely to encourage efficient counting strategies, though, because the second number is too large for many first graders to keep track of while counting.

Having noticed that number pairs cannot simultaneously encourage *counting* and *combining like units* strategies for children who only have a solid number sense with small numbers, the teachers chose to focus on *counting* strategies. The teachers then considered using a number greater than twenty and one less than ten. The combination (26, 3) was briefly proposed, but it did not yield many possibilities for different mental strategies. The combination (28, 3) was thought likely to encourage counting strategies. Three can be added incrementally to 28 (e.g., 28 + 2 = 30, 30 + 1 = 31), and those numbers might even prompt *compensation*-type strategies (e.g., 30 + 3 = 33, 33 – 2 = 31). For a greater challenge, the teachers decided to increase the magnitude of the second number. The number pair (28, 9) was the final choice for the lollipop problem. The numbers could be easily added by *direct modeling with ones, direct modeling with tens*, or even *counting* strategies. In addition, the problem could be solved by *incrementing* (28 + 2 = 30, 30 + 7 = 37) or *compensation*-type strategies. A student might think of adding 28 and 9 as both 30 + 9 = 39, 39 – 2 = 37 and as 28 + 10 = 38, 38 – 1 = 37.

The teachers were aware that this lesson addressed many different topics, but did not consider rushing the initial discussion section productive. Rather than doing so, so as to cram a whole-group discussion of student strategies for the lollipop problem into the end of the lesson, they concluded discussion of strategies used during the interview would offer enough opportunities for students to reflect on place value and extend their thinking within the time limits of the math segment of the day. The lollipop problem was intended to be a low-stakes opportunity for students to experiment with new mathematical ideas learned in the discussion in the first part of the lesson.

Lesson Plan

The account below took place in a first grade classroom. The lesson was developed in service of the following learning goals:

1. Students who direct modeled with ones will understand how groups of ten can be used in direct modeling strategies.
2. Students will use more efficient/sophisticated strategies.
3. Students' written records of their strategies will match the actual strategies they used.

The teacher said, "Do your remember the Pete's-rocks problem you solved earlier this morning?" She read the problem aloud: *Pete had 20 rocks. Juan gave him 24 more rocks. How many rocks does Pete have now?* She continued, "The other teachers and I saw some very interesting ways you thought about this problem. We saw such great thinking that I thought it important to share it here with everyone so we can learn from each other."

The teacher called Trinity to the board and asked her to solve the problem the same way she did during the interview, with physical manipulatives. Trinity brought her set of base ten manipulatives to the document camera and started to count out 20 single cubes. The teacher, realizing that Trinity may have forgotten the exact steps she took during the interview, said, "Hmm, the teacher that sat with when you first solved this problem told me you solved it differently." Trinity looked over her work on the interview paper and remembered that she had used base ten rods instead of cubes. She proceeded to take out two ten rods, placed then on one side of the camera surface, and then added two ten rods and four individual cubes nearby. She started explained, "I used two tens..." The teacher interrupted her to ask the entire group, "How much is two tens?" Several students answer in unison, "20!" Trinity counted out loud the two ten rods by ones to show it was 20. The teacher went to the white board, where Trinity's two ten rods were projected, and asked her peers if they can be counted another way. A student counted out loud, "10, 20." The teacher then wrote 10 and 20 to the right of the projection of each rod. The teacher asked Trinity to continue with her explanation. Trinity proceeded to count the combined sets starting with 20 and continuing the count by ones, "21, 22, 23, …" The teacher asked class if there was another way to count. Another student counted out loud, "20, 30, 40, 41, 42, 43, 44." The teacher wrote the numbers that corresponded to the rest of the ten rods and individual cubes to the right of the projection. She asked the student, "This is how you counted?" After he answered in the affirmative, the teacher thanked Trinity and the other students who helped and asked Trinity to return to her seat.

Next, Timothy was called to the front to share his solution for the Pete's-rocks problem. The teacher displayed the model he drew the first time he solved the problem (see Figure 2). Timothy explained, "I used the tools first, then I drew what I used. I took out 20 first. That was two sticks of ten. Then, I took out 24. That was two sticks of ten and four more. Then I counted and I got 24. I drew here what I used", pointing to the drawing on his worksheet. "I wrote ten in each row to show how much they were. The last row was four." The teacher noted, "I like how you circled the drawing for your first number and wrote down 20. And you did the same for 24, great. Your drawing shows me very well what you did. And how did you find your answer?" Timothy said, "I counted the rods together." The teacher asked Timothy, "And how did you count?" Timothy answered, "10, 20, 30, 40; 40 and four more was 44." The teacher next said to the entire class, "It's great how he didn't have to draw each cube for his picture. He just drew the rectangles." Next, the teacher asked, "Did any one of you remember using a different way to draw? A few students raised their hands. The teacher chose Madison to talk. Madison said, "I drew circles and counted each circle." The teacher asked, "How many circles did you draw?" Madison answered, "I drew 20 circles and then I drew 24 circles." The teacher replied, "Wow, that's a lot of circles to count!" She then asked the rest of the students, "How is Timothy's strategy different from Madison's?" One student replied, "Madison used circles and Timothy used the rods." The teacher prompted, "Any other differences?" The same student said, "The rods had ten together and Timothy counted faster that way." The teacher said, "I see." She asked Timothy to return to his seat.

The teacher asked Nicole to explain her strategy next. Nicole shared, "I kept 20 in my head and counted out 24 more." The teacher asked, "So you don't need to count out the 20? Can you just do that? Keep it in your head?" Nicole answered, "Yes, I know it's 20, so I start there." The teacher asked Nicole to count again using her strategy and asked the class to help Nicole count. Nicole said, "20"—pointing to her temple—and started counting using her fingers, "21 …". She extended her thumb of her left hand. The rest of the first graders counted in step with Nicole. The teacher copied Nicole's fingers' gestures and interjected, "That's my first finger…Continue." Nicole continued counting, extending her second finger on her left hand, "22 …". The teacher again interjected, "Second finger?" Teacher extended her second finger as well and nodded to Nicole to continue. Nicole counted, this time for longer, "23, 24, ..., 30." Her peers counted in unison with her. The teacher asked," How much did I add so far?" Nicole looked at her extended fingers and answered, "Ten." She was then encouraged to continue counting. Nicole closed her hands, then reopened them and restarted keeping track of counts with fingers, saying, "31, 32, 33, …, 40."

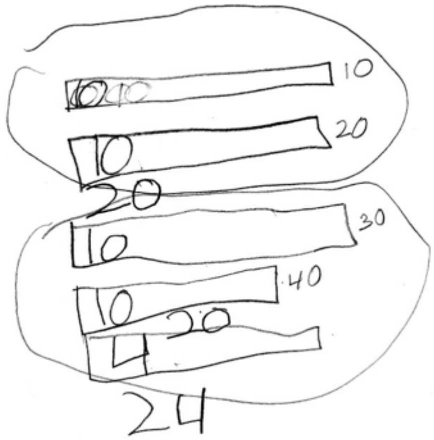

Figure 2. Timothy's direct modeling with tens strategy for the Pete's-rocks problem.

The teacher asked again, "So how much did we count on now?" Nicole answered, "20." The teacher asked her, "So how much do we have left to count?" Nicole said, "Four more." She proceeded to count keeping track on her fingers, "41, 42, 43, 44." The teacher said, "Wow, that was some impressive work. You were able to keep track of all those counts! But how can we record all this great thinking on paper so I can tell how you solved the problem even if you're not here to explain to us?" Nicole and her peers did not have an answer to this. After giving them several moments to think without any success, the teacher went to the board and said, "I think I can write the first number like this," she wrote 20 on the board and circled it. She then continued, "I can next draw my fingers and write the number they stand for on them." The teacher drew a quick illustration of the tip of a finger and wrote 21 on it, then repeated the process for several more numbers. She said, "And so on until you get to 44." The teacher wrote an ellipsis ("…") and finished the line drawing the last finger with 44 on it.

The teacher next asked Gianluca to explain his thinking. She displayed his student work for everyone to see (Figure 3). The teacher said, "I see a lot of numbers here. Where did they come from? Where did the 20 come from?" Gianluca answered, "Pete started with 20 rocks." The teacher asked again, "And where did the second 20 come from?" Gianluca said, "He got 24 more rocks. But I separated it into 20 and four because it's easier." The teacher asked Gianluca, "So, can you circle me where 24 is in your strategy so I can see it clearly?" Gianluca circled the second 20 and the four as seen in Figure 3. The teacher said, "So, what did you add first?" Gianluca answered, "The 20 and 20, they make 40. And then I added another four." The teacher asked, "and what do you get when you add 40 and four?" "44", Gianluca answered. The teacher turned to the class and said, "He used just numbers to find the answer. He didn't use any drawings or tools. Can he do that when solving a problem?" After thinking about it, some students said yes. "You're right," the teacher said. His numbers tell me exactly and clearly how he solved the problem."

The teacher called the last student to share, Catherine. The teacher said, "Catherine, the teachers told me that when you solved this problem you started with a number different from 20." Catherine answered, "Yes, 24." The teacher asked, "And then what did you do?" Catherine said, "I added ten and then ten more." The teacher wrote on the board, 24 + 10 →, and asked the class how much that was. One student answered "34." The teacher asked her, "How do you know?" The student counted on her fingers by one, "25, 26, 27, …, 34." The teacher wrote 34 to be the answer to the expression and continued, "And then Catherine added ten more. Class, how much is that?" A third student answered 44, then counted on by ones from 34 as evidence. The teacher asked the class one more time, "Why do you think Catherine added ten and then added ten more?" After thinking about the question, one student hesitantly answers, "Because she needed to add 20 and 24. She started with 24, so she needed to add 20 to it." The teacher asked the class, "Why do you think she started with 24? Talk it over with

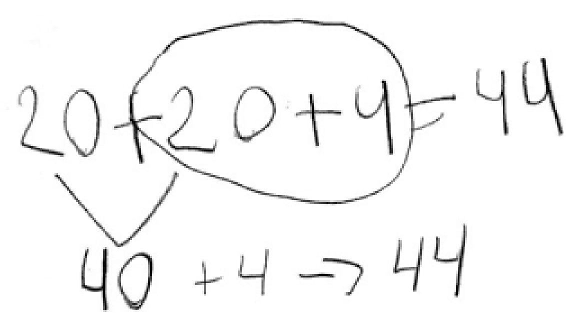

Figure 1. Gianluca's invented algorithm strategy for the Pete's-rocks problem.

Overall, seven students used a strategy to solve the lollipop problem that differed from the one they used in similar problems previously.

your table mates." The first graders took a few minutes to discuss their opinions about Catherine's reasoning while the teacher listened in on their conversations.

The teacher next gave the students the lollipop problem, "I have another problem for you to solve. You can use the base ten tools or fingers or solve it in your minds like Gianluca and Catherine. Whatever way you choose to use, I want you to write how you solved it down on your paper." The teacher handed out a sheet with the lollipop problem to each student. She then read the problem aloud, "*I had 28 lollipops. My friend gave me 9 more. How many lollipops do I have now?*" Students began working on the problem on their own. The teacher walked around the class and stopped to talk with various students about their strategies on how to find the answer. The students solved the problem and wrote their strategies down. If any student finished early, they were asked to solve the problem a different way.

After about ten minutes, most students were done. The teacher commended everyone for their thinking and ended the lesson.

Reflection

After the lesson concluded, the teachers gathered once more to discuss and reflect on it. They discussed the students' engagement, the dissemination of ideas, the different teaching strategies, and the overall success of meeting the goals they set for the lesson.

What we learned about the students

The students had been very engaged in the discussion. Although a lot of time was spent on *direct modeling* strategies, the teachers found them to have been useful in stimulating students' emerging concepts of ten. They appeared to notice and make sense of details in the organization of drawings and how to count objects. The result was a whole-class discussion about important concepts of multidigit numbers. The first graders were making connections between strategies that were thought to be valuable for deepening one's understanding of numbers. The extended sessions of counting in the class discussion had also been beneficial to students. Each number was counted by ones and by tens, sometimes more than one time. Even though counting is not often focused on in first grade, the students benefited from the purposeful engagement of counting the numbers. It also had the added benefit of engaging multiple students with someone else's strategy.

From the observations of how students solved the lollipop problem, the teachers saw evidence that many students had incorporated elements discussed during the lesson. Overall, seven students used a strategy to solve the lollipop problem that differed from the one they used in similar problems previously. Even though many students who had originally used *direct modeling* strategies still modeled, several students counted the tens in 28 by tens. That was considered a major breakthrough for those students. Some students used the base-ten rods to solve the problem. These students pulled out three rods and covered two units on the third rod to make 28. Finally, they added another rod or singles and counted on nine more.

Raina's thinking was particularly interesting to the teachers. She demonstrated a significant increase in her understanding of place value. She used base-ten manipulatives to solve the lollipop problem and wrote on her paper, 10 + 10 + 8 + 9. She added the tens and ones separately and found her answer by combining the partial sums. Her strategy was considered to be the precursor to an *invented algorithm* strategy, specifically for the *combining like units* strategy. When asked how she added the eight and the nine, Raina answered she added by ones. She was asked how she could find the answer if she did not have singles available. Raina answered that she would use two ten rods and cover two units on one and one unit on the other. While not pursued at that moment, the teachers saw the potential of a similar conversation to lead to a discussion around 8 + 9 = 20 – 3.

Two students who previously used a *counting by ones* strategy had drawn fingers with numbers on their papers to show their work, just like the teacher had done during the whole-class discussion. The teachers were happy that students were learning how to express their thinking processes on paper. On the other hand, the teachers noticed many discrepancies between how students had solved the lollipop problem and how they recorded their work on paper. Although they showed a more comfortable use of tens with physical tools, the students reverted back to models that were in line more with *direct modeling with ones* on paper. Some students had held the first number in their mind and only used manipulatives to add on the nine. When recording their work, however, they drew models for both numbers on paper. A teacher evaluating the students' thinking solely from the work on paper—as often happens—would miss the important milestone the students had reached. The teachers' conclusions were that more discussion is needed around clear and accurate ways students can show their thinking on paper.

Ideas for future lessons

On the basis of their observations, the teachers considered the future trajectory of the first graders' engagement with numbers. They agreed that students would probably need more opportunities to solve simple addition problem types—such as *join result unknown* or *part-part-whole*, *whole unknown* problems—with multidigit numbers. These simple problem types do not get in the way of thinking and encourage the use of more sophisticated strategies. Students would also benefit from the use of relatively small multidigit numbers initially to extend their understanding of place value. The teachers concluded that many more conversations about how to record their thinking accurately on paper would be needed.

The teachers observed that the purposeful selection of numbers to use in problems posed to students is a crucial part of careful planning for instruction. As an extension to what happened in the lesson, they could have reversed the order in which the numbers were presented as a way to prompt students to discuss the merits of starting their counting from the greater number. For example, Pete might have nine lollipops and receive 28 more from his friend. Student strategies that start with the 28 suggest that students may have an intuitive understanding of the commutative property of addition, and their

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ideas could provoke a discussion about whether would always be a valid strategy (and under what conditions it might not be a valid strategy). Using decade numbers (e.g., 20, 30) may help students to begin using *incrementing* strategies based on place-value concepts. Nondecade numbers (e.g., 24 and 37) can be used to encourage *combining like units* strategies. Nondecade numbers close to decade numbers (e.g. 29 and 19) may offer a way to introduce *compensation* strategies to the class repertoire. The teachers spent approximately 45 minutes discussing the potential numbers to use in the lollipop problem, and that careful deliberation and planning may have contributed to the success of the lesson.

What's Next?

Stories of teachers engaging in collaborative inquiry focused on using student thinking to inform instructional decisions

What's Next? is a collection of stories documenting professional development experiences shared by elementary teachers working collaboratively to study the complex process of teaching and learning mathematics. Each story in the collection describes practicing teachers studying the thinking processes of real students and using what they learn about those students to make decisions and try to help advance those students' understanding on that day.

The teachers in each story start by learning about how individual students are solving a set of mathematics problems. They use this freshly gathered knowledge of student thinking to develop near-term learning goals for students and a lesson plan tailored to specific students on that specific day. One of the teachers implements the planned lesson while the other teachers observe in real time. The teachers then gather to discuss and reflect on their observations and insights.

In these lessons, the practice of teaching is slowed way down. The stories tell of teachers who are studying student thinking and using that information to plan and implement instructional decisions at a pace that is much slower than it occurs in daily practice. The stories in this collection also depict many aspects in common with formative assessment and lesson study, both of which are a process and not an outcome.

The stories depict real situations that occurred in real time and include both successes and shortcomings. We hope that the stories may be studied and discussed by interested educators so that the lessons and ideas experiences of these teachers and instructional coaches may contribute to additional learning and sharing among other interested teachers.

Learn more about these and other stories at **http://www.teachingisproblemsolving.org/**